Parametrized modal logic: the unidimensional case

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1. **Modal logics with intersection**
2. Unidimensional parametrized modal logic

Modal logics with intersection

Ordinary modal logics

Syntax

- $\varphi \ := \ p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Diamond\varphi$

Models : $M \ = \ (W, R, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- R binary relation on W
- $V : p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \Diamond\varphi$ iff $\exists t$ (sRt and $t \models \varphi$)

Modal logics with intersection

Modal logics with an algebraic structure in modalities

Syntax

- $\alpha \ := \ a \mid (\alpha \cup \alpha) \mid (\alpha \cap \alpha)$
- $\varphi \ := \ p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\alpha\rangle\varphi$

Models : $M \ = \ (W, \mathbf{R}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- $\mathbf{R} : \alpha \mapsto \mathbf{R}(\alpha)$ binary relation on W such that
  - $\mathbf{R}(\alpha \cup \beta) = \mathbf{R}(\alpha) \cup \mathbf{R}(\beta)$
  - $\mathbf{R}(\alpha \cap \beta) = \mathbf{R}(\alpha) \cap \mathbf{R}(\beta)$
- $V : p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \langle\alpha\rangle\varphi$ iff $\exists t \ (s\mathbf{R}(\alpha)t$ and $t \models \varphi)$

Modal logics with intersection

Modal logics with an algebraic structure in modalities

There is no problem with $\langle \alpha \cup \beta \rangle$

$$
\begin{aligned}
& s \models \langle \alpha \cup \beta \rangle \varphi \\
\Leftrightarrow\ & \exists t\ (s\mathbf{R}(\alpha \cup \beta)t \text{ and } t \models \varphi) \\
\Leftrightarrow\ & \exists t\ ((s\mathbf{R}(\alpha)t \text{ or } s\mathbf{R}(\beta)t) \text{ and } t \models \varphi) \\
\Leftrightarrow\ & \exists t\ ((s\mathbf{R}(\alpha)t \text{ and } t \models \varphi) \text{ or } (s\mathbf{R}(\beta)t \text{ and } t \models \varphi)) \\
\Leftrightarrow\ & \exists t\ (s\mathbf{R}(\alpha)t \text{ and } t \models \varphi) \text{ or } \exists t\ (s\mathbf{R}(\beta)t \text{ and } t \models \varphi)) \\
\Leftrightarrow\ & s \models \langle \alpha \rangle \varphi \text{ or } s \models \langle \beta \rangle \varphi \\
\Leftrightarrow\ & s \models \langle \alpha \rangle \varphi \vee \langle \beta \rangle \varphi
\end{aligned}
$$

Axiomatization will contain

$\langle \alpha \cup \beta \rangle p \leftrightarrow \langle \alpha \rangle p \vee \langle \beta \rangle p$

Modal logics with intersection

Modal logics with an algebraic structure in modalities

There is a problem with $\langle \alpha \cap \beta \rangle$

$$\begin{aligned}
& s \models \langle \alpha \cap \beta \rangle \varphi \\
\Leftrightarrow\ & \exists t\ (s\mathbf{R}(\alpha \cap \beta)t \text{ and } t \models \varphi) \\
\Leftrightarrow\ & \exists t\ ((s\mathbf{R}(\alpha)t \text{ and } s\mathbf{R}(\beta)t) \text{ and } t \models \varphi) \\
\Leftrightarrow\ & \exists t\ ((s\mathbf{R}(\alpha)t \text{ and } t \models \varphi) \text{ and } (s\mathbf{R}(\beta)t \text{ and } t \models \varphi)) \\
\Rightarrow\ & \exists t\ (s\mathbf{R}(\alpha)t \text{ and } t \models \varphi) \text{ and } \exists t\ (s\mathbf{R}(\beta)t \text{ and } t \models \varphi)) \\
\Leftrightarrow\ & s \models \langle \alpha \rangle \varphi \text{ and } s \models \langle \beta \rangle \varphi \\
\Leftrightarrow\ & s \models \langle \alpha \rangle \varphi \wedge \langle \beta \rangle \varphi
\end{aligned}$$

Axiomatization will contain

$\langle \alpha \cap \beta \rangle p \rightarrow \langle \alpha \rangle p \wedge \langle \beta \rangle p$

Modal logics with intersection

BML : Boolean Modal Logic

Syntax

- $\alpha \ := \ a \mid 1 \mid \bar{\alpha} \mid (\alpha \cup \alpha)$
- $\varphi \ := \ p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\alpha\rangle\varphi$

Models : $M \ = \ (W, \mathbf{R}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- $\mathbf{R} : \alpha \mapsto \mathbf{R}(\alpha)$ binary relation on W such that
  - $\mathbf{R}(1) = W \times W$
  - $\mathbf{R}(\bar{\alpha}) = W \times W \setminus \mathbf{R}(\alpha)$
  - $\mathbf{R}(\alpha \cup \beta) = \mathbf{R}(\alpha) \cup \mathbf{R}(\beta)$
- $V : p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \langle\alpha\rangle\varphi$ iff $\exists t \ (s\mathbf{R}(\alpha)t$ and $t \models \varphi)$

Modal logics with intersection

DAL : Data Analysis Logic

Syntax

- $\alpha \;:=\; a \mid (\alpha \cap \alpha) \mid (\alpha \cup \alpha)^+$
- $\varphi \;:=\; p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\alpha\rangle\varphi$

Models : $M \;=\; (W, \mathbf{R}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- $\mathbf{R} : \alpha \mapsto \mathbf{R}(\alpha)$ equivalence relation on W such that
  - $\mathbf{R}(\alpha \cap \beta) = \mathbf{R}(\alpha) \cap \mathbf{R}(\beta)$
  - $\mathbf{R}((\alpha \cup \beta)^+) = (\mathbf{R}(\alpha) \cup \mathbf{R}(\beta))^+$
- $V : p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \langle\alpha\rangle\varphi$ iff $\forall t$ ($s\mathbf{R}(\alpha)t$ and $t \models \varphi$)

Modal logics with intersection

PDL with intersection

Syntax

- $\alpha \ := \ a \mid (\alpha;\alpha) \mid (\alpha \cup \alpha) \mid \alpha^\star \mid \varphi? \mid (\alpha \cap \alpha)$
- $\varphi \ := \ p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\alpha\rangle\varphi$

Models : $M \ = \ (W, \mathbf{R}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- $\mathbf{R} \,:\, \alpha \mapsto \mathbf{R}(\alpha)$ binary relation on W such that
  - $\mathbf{R}(\alpha;\beta) = \mathbf{R}(\alpha) \circ \mathbf{R}(\beta)$
  - $\mathbf{R}(\alpha \cup \beta) = \mathbf{R}(\alpha) \cup \mathbf{R}(\beta)$
  - $\mathbf{R}(\alpha^\star) = \mathbf{R}(\alpha)^\star$
  - $\mathbf{R}(\alpha \cap \beta) = \mathbf{R}(\alpha) \cap \mathbf{R}(\beta)$
- $V \,:\, p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \langle\alpha\rangle\varphi$ iff $\exists t \ (s\mathbf{R}(\alpha)t$ and $t \models \varphi)$

Modal logics with intersection

$\mathbf{S5}_n^D$: Epistemic Logic with Distributed Knowledge

Syntax

- $\varphi \; := \; p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle G \rangle \varphi$

where $G \in \wp(\mathcal{A})$ for some fixed finite nonempty set $\mathcal{A}$ of "agents"

Models : $M \; = \; (W, \mathbf{R}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set of "worlds"
- $\mathbf{R} : \wp(\mathcal{A}) \longrightarrow \wp(W \times W)$ is such that for all $G \in \wp(\mathcal{A})$
  - $\mathbf{R}(G) = \bigcap\{\mathbf{R}(\{i\}) : \; i \in G\}$ is an equivalence relation on W
- $V : p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \langle G \rangle \varphi$ iff $\exists t \; (s\mathbf{R}(G)t$ and $t \models \varphi)$

1. Modal logics with intersection
2. **Unidimensional parametrized modal logic**

Unidimensional parametrized modal logic (UPML)

Ordinary modal logics

Syntax

- $\varphi \ := \ p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Diamond\varphi$

Models : $M \ = \ (W, R, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- R binary relation on W
- $V : p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \Diamond\varphi$ iff $\exists t$ (sRt and $t \models \varphi$)

Unidimensional parametrized modal logic (UPML)

Ordinary modal logics with a $\Diamond$ of arity 2

Syntax

- $\varphi \;:=\; p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \Diamond \varphi)$

Models : $M \;=\; (W, R, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- R ternary relation on W
- $V \,:\, p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \varphi \Diamond \psi$ iff $\exists u \; (\exists t \; (t \models \varphi$ and $sR(t, u))$ and $u \models \psi)$

Unidimensional parametrized modal logic (UPML)

New modal logics with a ⧫ of arity 2

Syntax

- $\varphi \;:=\; p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \blacklozenge \varphi)$

Models : $M \;=\; (W, R, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- R ternary relation on W
- $V \,:\, p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \varphi \blacklozenge \psi$ iff $\exists u \; (\forall t \; (t \models \varphi \Rightarrow sR(t, u))$ and $u \models \psi)$

Unidimensional parametrized modal logic (UPML)

From $R \subseteq W \times W \times W$ to $\mathbf{R}: \wp(W) \longrightarrow \wp(W \times W)$

For all $A \in \wp(W)$ and for every $s, u \in W$

- $s\mathbf{R}(A)u$ exactly when $\forall t \ (t \in A \Rightarrow sR(t,u))$

An important property

For all $A \in \wp(W)$

- $\mathbf{R}(A) = \bigcap\{\mathbf{R}(\{t\}) : \ t \in A\}$

Other properties

For all $A, B \in \wp(W)$

- $\mathbf{R}(\emptyset) = W \times W$
- if $A \subseteq B$ then $\mathbf{R}(A) \supseteq \mathbf{R}(B)$

Unidimensional parametrized modal logic (UPML)

New modal logics with a $\blacklozenge$ of arity 2

Syntax

- $\varphi \;:=\; p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi\blacklozenge\varphi)$

Models : $M \;=\; (W, \mathbf{R}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- $\mathbf{R} : \; \wp(W) \longrightarrow \wp(W \times W)$ such that for all $A \in \wp(W)$
  - $\mathbf{R}(A) = \bigcap\{\mathbf{R}(\{t\}) : \; t \in A\}$
- $V \,:\, p \mapsto V(p)$ subset of W

Some of the truth-conditions

- $s \models \varphi\blacklozenge\psi$ iff $\exists u \; (s\mathbf{R}(V(\varphi))u$ and $u \models \psi)$

Abbreviation

- $(\varphi\blacksquare\psi) \;:=\; \neg(\varphi\blacklozenge\neg\psi)$

Corresponding truth-condition

- $s \models \varphi\blacksquare\psi$ iff $\forall u \; (s\mathbf{R}(V(\varphi))u \Rightarrow u \models \psi)$

Unidimensional parametrized modal logic (UPML)

New modal logics with a ♦ of arity 2

Syntax

- $\varphi \ := \ p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\varphi\rangle\varphi$

Models : $M \ = \ (W, \mathbf{R}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- $\mathbf{R} : \ \wp(W) \longrightarrow \wp(W \times W)$ such that for all $A \in \wp(W)$
  - $\mathbf{R}(A) = \bigcap\{\mathbf{R}(\{t\}) : \ t \in A\}$
- $V \ : \ p \mapsto \ V(p)$ subset of W

Some of the truth-conditions

- $s \models \langle\varphi\rangle\psi$ iff $\exists u \ (s\mathbf{R}(V(\varphi))u$ and $u \models \psi)$

Abbreviation

- $[\varphi]\psi \ := \ \neg\langle\varphi\rangle\neg\psi$

Corresponding truth-condition

- $s \models [\varphi]\psi$ iff $\forall u \ (s\mathbf{R}(V(\varphi))u \Rightarrow u \models \psi)$

Unidimensional parametrized modal logic (UPML)

Frames

Structures of the form $(W, \mathbf{R})$ where

- $W = \{s, t, \ldots\}$ is a nonempty set
- $\mathbf{R} : \ \wp(W) \longrightarrow \wp(W \times W)$

Conjunctive frames

Frames $(W, \mathbf{R})$ such that for all $A \in \wp(W)$

- $\mathbf{R}(A) = \bigcap\{\mathbf{R}(\{t\}) : \ t \in A\}$

Paraconjunctive frames

Frames $(W, \mathbf{R})$ such that $\mathbf{R}(\emptyset) = W \times W$ and for all $A, B \in \wp(W)$

- if $A \subseteq B$ then $\mathbf{R}(A) \supseteq \mathbf{R}(B)$

Unidimensional parametrized modal logic (UPML)

Valid formulas

- tautologies
- $[p](q \to r) \to ([p]q \to [p]r)$
- $[\bot]p \to p$
- $\langle\bot\rangle p \to [\bot]\langle\bot\rangle p$
- $[\bot](p \to q) \to ([p]r \to [q]r)$

Admissible rules

- $\frac{p,\ p \to q}{q}$
- $\frac{p}{[q]p}$
- $\frac{p \leftrightarrow q}{[p]r \leftrightarrow [q]r}$

Axiomatization

- let $\mathbf{K_c}$ be the calculus consisting of the above axioms and rules

Unidimensional parametrized modal logic (UPML)

Completeness in the class of all paraconjunctive frames

- $(W_c, \mathbf{R}_c, V_c)$ is paraconjunctive: $\mathbf{R}_c(\emptyset) = W_c \times W_c$ and for all $A, B \in \wp(W_c)$
  - if $A \subseteq B$ then $\mathbf{R}_c(A) \supseteq \mathbf{R}_c(B)$
- Truth Lemma: for all formulas φ and for all $s \in W_c$
  - $s \models \varphi$ if and only if $\varphi \in s$
- $\mathbf{K_c}$ exactly axiomatizes validities in the class of all paraconjunctive frames

Completeness in the class of all conjunctive frames

- every paraconjunctive frame is a bounded morphic image of a conjunctive frame
- $\mathbf{K_c}$ exactly axiomatizes validities in the class of all conjunctive frames

Conclusion

What has been done ?

- syntax of UPMLs
- semantics
- canonical model construction
- copying construction
- filtration method

Conclusion

What can be done ?

- import first-order ideas into UPMLs
- develop the model theory of UPMLs
- elaborate the correspondence theory of UPMLs
- investigate the computability of the satisfiability problem in such-and-such class of conjunctive frames and develop automatic procedures for solving it
- compare UPMLs with other forms of modal logics based on parametrized connectives
- construct the duality theory of UPMLs
- relationships with
  - Shi, C., Sun, Y. *Logic of convex order.* Studia Logica **109** (2021) 1019–1047.

Conclusion

Traditional syntax of EL with Distributed Knowledge

- Formulas: $\langle G \rangle \varphi$

where $G \in \wp(\mathcal{A})$ for some fixed finite nonempty set $\mathcal{A}$ of "agents"

Traditional models of EL with Distributed Knowledge

$M \; = \; (W, \mathbf{R}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set of "worlds"
- $\mathbf{R} : \wp(\mathcal{A}) \longrightarrow \wp(W \times W)$ is such that for all $G \in \wp(\mathcal{A})$
  - $\mathbf{R}(G) = \bigcap \{\mathbf{R}(\{i\}) : \; i \in G\}$ is an equivalence relation on W

Some of the traditional truth-conditions of EL with Distributed Knowledge

- $s \models \langle G \rangle \varphi$ iff $\exists t \; (s\mathbf{R}(G)t$ and $t \models \varphi)$

Conclusion

What more ? Bidimensional parametrized modal logic

– Syntax

- $\varphi \ := \ p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\alpha\rangle\varphi$ — "world formulas"
- $\alpha \ := \ a \mid \top \mid \neg\alpha \mid (\alpha \wedge \alpha) \mid \langle\varphi\rangle\alpha$ — "agent formulas"

– Models : $M \ = \ (W, \mathcal{A}, \mathbf{R}, \mathbf{F}, V)$ where

- $W = \{s, t, \ldots\}$ is a nonempty set of "worlds"
- $\mathcal{A} = \{i, j, \ldots\}$ is a nonempty set of "agents"
- $\mathbf{R} : \ \wp(\mathcal{A}) \longrightarrow \wp(W \times W)$
- $\mathbf{F} : \ \wp(W) \longrightarrow \wp(\mathcal{A} \times \mathcal{A})$
- $V \ : \ p \mapsto V(p)$ subset of W and $a \mapsto V(a)$ subset of $\mathcal{A}$

Some of the truth-conditions

- $s \models \langle\alpha\rangle\varphi$ iff $\exists t \in W \ (s\mathbf{R}(V(\alpha))t$ and $t \models \varphi)$
- $i \models \langle\varphi\rangle\alpha$ iff $\exists j \in \mathcal{A} \ (i\mathbf{F}(V(\varphi))j$ and $j \models \alpha)$

Thank you